

B.E.2nd Sem. Examination, 2008-09
Paper : Math-102 - E

Note:- Attempt **five** questions in all, selecting at least **one** question from each section.

SECTION-A

1. (a) Find the rank of the matrix : $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ by reducing it to the normal form. 10

Solution. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

operating $R_3 \rightarrow R_3 + R_1, R_2 \rightarrow R_2 + 2R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

operating $C_1 \rightarrow \frac{1}{3}C_1, C_2 \rightarrow -C_2, C_3 \rightarrow \frac{1}{2}C_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

operating $C_3 \rightarrow C_3 - C_1, C_2 \rightarrow C_2 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

operating $R_2 \rightarrow \frac{1}{4}R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

operating $C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & : & 0 \\ \dots & & \dots \\ 0 & : & 0 \end{bmatrix}$$

which is the required Normal form $\therefore \rho(A) = 2$ **Ans.**

1. (b) Compute the inverse of $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ using elementary transformations. 10

Solution. Writing the given matrix side by side with unit Matrix I_3 , we get

$$[A : I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array} \right]$$

operating $R_1 \rightarrow \frac{1}{2}R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array} \right]$$

operating $R_3 \rightarrow R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 & 1 \end{array} \right]$$

operating $R_2 \rightarrow \frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 & 1 \end{array} \right]$$

operating $R_1 \rightarrow R_1 - \frac{1}{2}R_2$ and $R_3 \rightarrow R_3 + \frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{5}{2} & \frac{1}{4} & 1 \end{array} \right]$$

operating $R_3 \rightarrow 4R_3$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{3}{4} & : & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & : & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & : & 10 & -1 & -4 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - \frac{1}{2}R_3$ and $R_1 \rightarrow R_1 + \frac{3}{4}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 8 & -1 & -3 \\ 0 & 1 & 0 & : & -5 & 1 & 2 \\ 0 & 0 & 1 & : & 10 & -1 & -4 \end{bmatrix}$$

$$= [I_3 : A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

Ans.

2. (a) Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

and hence compute A^{-1}

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Solution. The characteristic equation of A is

$$|A - \lambda I| = 0 \text{ i.e., } \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \text{ (on simplification)}$$

To verify Cayley - Hamilton theorem, we have to show that

$$A^2 - 4A^3 - 20A - 35I = 0$$

....(1)

Now

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\therefore A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$-20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley - Hamilton theorem.

Now, multiplying both sides of (1) by A^{-1} , we have

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

or

$$35A^{-1} = A^2 - 4A - 20I$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$. **Ans.**

2. (b) Show that the transformation $y_1 = \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3$

$$y_2 = \frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3$$

$$y_3 = \frac{2}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 \text{ is orthogonal.}$$

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Solution. In Matrix notation, the given transformation $Y = AX$,

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\therefore A' = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now $AA' = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Since $A A' = 1$, A is an orthogonal Matrix.
 \therefore Given transformation is orthogonal. **Proved.**

SECTION - B

3. (a) Solve the D.E. $(y \cos x + 1) dx + \sin x dy = 0$

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Solution. Here $M = (y \cos x + 1)$ and $N = \sin x$

$$\therefore \frac{\partial M}{\partial y} = \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the given diff. eq. is exact and its solution is

$$\int_{y \text{ const}} M dx + \int [\text{terms } N \text{ not containing } x] dy = C$$

$$\int_{y \text{ const}} (y \cos x + 1) dx + \int 0 dy = C$$

$$y \sin x + x = c. \quad \text{Ans.}$$

3 (b) Solve the D.E. $x dy - y dx = (x^2 + y^2) dx$.

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Solution.

$$x dy - y dx = (x^2 + y^2) dx$$

or

$$\frac{x dy - y dx}{x^2 + y^2} = dx$$

or

$$d \left[\tan^{-1} \left(\frac{y}{x} \right) \right] = dx$$

$$\text{On Integrating} \quad \tan^{-1} \left(\frac{y}{x} \right) = x + c$$

$$\left(\frac{y}{x} \right) = \tan(x + c)$$

$$y = x \tan(x + c). \quad \text{Ans.}$$

3. (c) If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes. 7

Ans. By Newton's law of cooling

$$\frac{dT}{dt} = -k(T - 30) \quad \text{where } k \text{ is a constant.}$$

or

$$\frac{dT}{(T - 30)} = -k dt$$

On Integrating

$$\log(T - 30) = -kt + c$$

....(i)

But at

$$t = 0, \quad T = 80^\circ\text{C}$$

\therefore

$$\log(80 - 30) = -k \times 0 + c \Rightarrow C = \log 50$$

From (i)

$$\log(T - 30) = -kt + \log 50$$

$$\begin{aligned}
 & kt = \log 50 - \log(T - 30) \quad \dots(ii) \\
 \text{when } T = 60^\circ, t = 12 \\
 & 12k = \log 50 - \log(60^\circ - 30^\circ) \\
 & 12k = \log 50 - \log 30 \quad \dots(iii)
 \end{aligned}$$

Dividing (ii) by (iii)

$$\frac{kt}{12k} = \frac{\log \left[\frac{50}{T-30} \right]}{\log \left[\frac{50}{30} \right]}$$

$$\frac{t}{12} = \frac{\log \left[\frac{50}{T-30} \right]}{\log \left[\frac{5}{3} \right]}$$

At $t = 24$

$$\frac{24}{12} = \frac{\log \left[\frac{50}{T-30} \right]}{\log \left[\frac{5}{3} \right]}$$

or

$$2 \log \left(\frac{5}{3} \right) = \log \left[\frac{50}{T-30} \right]$$

$$\log \left(\frac{5}{3} \right)^2 = \log \left[\frac{50}{T-30} \right]$$

$$\frac{25}{9} = \frac{50}{T-30}$$

or

$$T - 30 = 18$$

\therefore

$$T = 18 + 30 = 48^\circ$$

Hence the temperature will be 48°C after 24 minutes. **Ans.**

4. (a) Solve the D.E. $\frac{d^2 y}{dx^2} + 4y = x \sin x$

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Solution. Given equation. in symbolic form is

$$(D^2 + 4) y = x \sin x$$

$$\text{A. E. is } D^2 + 4 = 0$$

$$D^2 = -4 \Rightarrow D = \pm 2i$$

\therefore

$$\text{C.F.} = C_1 \cos 2x + C_2 \sin 2x.$$

$$\text{P.I} = \frac{1}{D^2 + 4} (x \sin x)$$

$$= \text{I.P. of } \frac{1}{(D^2 + 4)} x[\cos x + i \sin x]$$

$$= \text{I.P. of } \frac{1}{(D^2 + 4)} x e^{ix}$$

$$[\because e^{ix} = \cos x + i \sin x]$$

$$\begin{aligned}
&= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2 + 4} (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{D^2 + i^2 + 2iD + 4} (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{D^2 - 1 + 2iD + 4} (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{(D^2 + 2iD + 3)} (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{3 \left[1 + \frac{2iD}{3} + \frac{D^2}{3} \right]} (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{3} \left[1 + \frac{2iD}{3} + \frac{D^2}{3} \right]^{-1} (x) \quad [\because (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots \text{to } \infty] \\
&= \text{I.P. of } e^{ix} \frac{1}{3} \left[1 - \left(\frac{2iD}{3} + \frac{D^2}{3} \right) + \left(\frac{2iD}{3} + \frac{D^2}{3} \right)^2 \dots \right] (x) \\
&= \text{I.P. of } e^{ix} \frac{1}{3} \left[x - \frac{2iD}{3} (x) \right] = \text{I.P. of } e^{ix} \frac{1}{3} \left[1 - \frac{2i}{3} \right] \\
&= \text{I.P. of } \frac{1}{3} [\cos x + i \sin x] \left[x - \frac{2i}{3} \right] = \frac{-2}{9} \cos x + \frac{x}{3} \sin x \\
&= \frac{1}{9} [3x \sin x - 2 \cos x]
\end{aligned}$$

Hence the C.S. is $y = C.F + P.I.$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{9} [3x \sin x - 2 \cos x]. \quad \text{Ans.}$$

4. (b) Solve the D.E. $\frac{d^2 y}{dx^2} + y = \tan x$ by the method of variation of parameters. 10

Solution. Given equation in symbolic form is $(D^2 + 1) y = \tan x$

A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

\therefore C.F. = $C_1 \cos x + C_2 \sin x$

Here $y_1 = \cos x$, $y_2 = \sin x$ and $X = \tan x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos x \int \sin x \tan x dx + \sin x \int \cos x \tan x dx$$

$$\begin{aligned}
&= -\cos x \int \frac{(1 - \cos^2 x)}{\cos x} dx + \sin x \int \sin x dx \\
&= -\cos x \int (\sec x - \cos x) dx + \sin x \times -\cos x \\
&= -\cos x [\log (\sec x + \tan x) - \sin x] - \sin x \cos x \\
&= -\cos x \log (\sec x + \tan x) + \cos x \sin x - \sin x \cos x \\
&= -\cos x \log (\sec x + \tan x)
\end{aligned}$$

Hence C.S. is $y = C_1 \cos x + C_2 \sin x - \cos x \log (\sec x + \tan x)$. **Ans.**

5. (a) Solve the D.E. $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x.$

10

Solution. This is a Cauchy's linear homogeneous eq.

Put $x = e^z$ i.e. $z = \log x$, so that

$$\frac{x dy}{dx} = Dy, x^2 \frac{dy^2}{dx^2} = D(D-1)y, \text{ where } D = \frac{d}{dz}$$

Then the given equation becomes

$$[D(D-1) - 2D - 4]y = (e^z)^2 + 2z$$

$$[D^2 - D - 2D - 4]y = e^{2z} + 2z$$

$$[D^2 - 3D - 4] = e^{2z} + 2z$$

Which is a linear eq. with constant co-efficients.

Its A.E. is

$$D^2 - 3D - 4 = 0$$

$$D^2 - 4D + D - 4 = 0$$

$$D(D-4) + 1(D-4) = 0$$

$$(D+1)(D-4) = 0$$

\therefore

$$D = -1, 4$$

Then

$$\text{C.F.} = C_1 e^{-z} + C_2 e^{4z}$$

$$= \frac{C_1}{e^z} + C_2 (e^z)^4$$

$$= \frac{C_1}{x} + C_2 x^4$$

$$\text{P.I.} = \frac{e^{2z} + 2z}{(D^2 - 3D - 4)}$$

$$= \frac{e^{2z}}{(D^2 - 3D - 4)} + \frac{e^{2z}}{(D^2 - 3D - 4)}$$

$$= \frac{e^{2z}}{(2)^2 - 3 \times 2 - 4} + \frac{2}{-4 - 3D + D^2} (z)$$

Now

$$= \frac{e^{2z}}{4 - 6 - 4} - \frac{2}{4 \left[1 + \frac{3}{4} D - \frac{D^2}{4} \right]} (z)$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2} \left[1 + \left(\frac{3D}{4} - \frac{D^2}{4} \right) \right]^{-1} (z)$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2} \left[1 - \frac{3D}{4} \dots \dots \right] z$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2} \left[z - \frac{3D}{4} (z) \right]$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2} z + \frac{3}{8} \times 1$$

$$= -\frac{3}{8} - \frac{e^{2z}}{6} - \frac{1}{2} z$$

$$= -\frac{3}{8} - \frac{x^2}{6} - \frac{1}{2} \log x \quad [\because e^z = x \text{ or } z = \log x]$$

Hence C.S. is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 x^{-1} + C_2 x^4 + \frac{3}{8} - \frac{x^2}{6} - \frac{1}{2} \log x. \text{ Ans.}$$

Q. 5 (b) Solve the simultaneous eqn. : $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ given that $x = 2$, $y = 0$ when $t = 0$

10

Solution. Here, $\frac{dx}{dt} + y = \sin t$

$$\text{or} \quad dx + y = \sin t \quad \dots (i)$$

$$\frac{dy}{dt} + x = \cos t$$

$$Dy + x = \cos t \quad \text{or} \quad x + Dy = \cos t \quad \dots (ii)$$

$$D^2 x + Dy = D[\sin t]$$

$$x + Dy = \cos t$$

— — — on subtracting

$$(D^2 - 1)x = \cos t - \cos t$$

$$\text{or} \quad (D^2 - 1)x = 0$$

$$\text{A.E. is } D = \pm 1$$

$$\therefore \text{C.F.} = C_1 e^{-t} + C_2 e^t$$

$$\text{P.I.} = 0$$

$$\text{Then } x = \text{C.F.} + \text{P.I.} = C_1 e^{-t} + C_2 e^t \quad \dots (iii)$$

Differentiating w.r.t. x

$$\frac{dx}{dt} = -C_1 e^{-t} + C_2 e^t$$

But

$$\frac{dx}{dt} + y = \sin t$$

$$-C_1 e^{-t} + C_2 e^t + y = \sin t \Rightarrow y = \sin t + C_1 e^{-t} - C_2 e^t \quad \dots (iv)$$

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But $x = 2$ at $t = 0$

From (iii) $2 = C_1 + C_2$ or $C_1 + C_2 = 2$ (v)

$y = 0$ at $t = 0$, $0 = \sin 0 + C_1 - C_2$ or $C_1 - C_2 = 0$ (vi)

From (v) and (vi), we get

$$C_1 = 1, C_2 = 1$$

Then $x = e^{-t} + e^t$
 $y = \sin t + e^{-t} - e^t$ **Ans.**

SECTION - C

6. (a) Find the Laplace transform of $\cosh(at) \sin(at)$

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Solution. Here, $L[\cosh(at) \sin(at)]$

$$\begin{aligned} & L\left[\left\{\frac{e^{at} + e^{-at}}{2}\right\} \sin at\right] \quad \left\{\because \cosh(at) = \frac{e^{at} + e^{-at}}{2}\right\} \\ &= \frac{1}{2} [L\{e^{at} \sin at\} + L\{e^{-at} \sin at\}] \\ &= \frac{1}{2} \left[\frac{a}{(s-a)^2 + a^2} + \frac{a}{(s+a)^2 + a^2} \right] \\ &= \frac{a}{2} \left[\frac{1}{s^2 + a^2 - 2sa + a^2} + \frac{1}{s^2 + a^2 + 2sa + a^2} \right] \\ &= \frac{a}{2} \left[\frac{1}{(s^2 + 2a^2) - (2sa)} + \frac{1}{(s^2 + a^2) + (2sa)} \right] \\ &= \frac{a}{2} \left[\frac{s^2 + 2a^2 + 2sa + s^2 + 2a^2 - 2sa}{\{(s^2 + 2a^2) - (2sa)\} \{(s^2 + 2a^2) + (2sa)\}} \right] \\ &= \frac{a}{2} \times \left[\frac{2s^2 + 4a^2}{(s^2 + 2a^2)^2 - (2sa)^2} \right] \\ &= \frac{a(s^2 + 2a^2)}{s^4 + 4a^4 + 4s^2a^2 - 4s^2a^2} = \frac{a(s^2 + 2a^2)}{s^4 + 4a^4} \quad \text{Ans.} \end{aligned}$$

6. (b) Find the Inverse Laplace transform of $\frac{1}{s^2(s^2 + a^2)}$.

7

Solution. Since the fraction involves only even powers of s , We put $s^2 = p$ so that the given fraction becomes

$$\frac{1}{p(p + a^2)}$$

Let $\frac{1}{p(p + a^2)} = \frac{A}{p} + \frac{B}{p + a^2}$

Multiplying both sides by $p(p+a^2)$, we get

$$1 = A(p + a^2) + pB$$

or

$$1 = Ap + Aa^2 + pB \quad \text{or} \quad 1 = (A+B)p + Aa^2$$

Equating Co-efficients of p , $0 = A + B$

...(i)

Equating constant terms $1 = Aa^2$

$$\text{or} \quad A = \frac{1}{a^2}$$

From (i) $B = -A = \frac{-1}{a^2}$

Thus $\frac{1}{s^2(s^2 + a^2)} = \frac{1}{p(p + a^2)} = \frac{1}{a^2 s^2} - \frac{1}{a^2(s^2 + a^2)}$

$$\therefore L^{-1} \left[\frac{1}{s^2(s^2 + a^2)} \right] = \frac{1}{a^2} L^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{a^2} L^{-1} \left[\frac{1}{s^2 + a^2} \right]$$

$$= \frac{1}{a^2} \frac{t^{2-1}}{(2-1)!} - \frac{1}{a^2} \times \frac{1}{a} \sin at$$

$$= \frac{t}{a^2} - \frac{1}{a^3} \sin at = \frac{1}{a^3} [at - \sin at]. \quad \text{Ans.}$$

6. (c) Evaluate : $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt.$

6

Solution. Here, $L[e^{-t} - e^{-3t}] = \frac{1}{s+1} - \frac{1}{s+3}$

$$\therefore L \left[\frac{e^{-t} - e^{-3t}}{t} \right] = \int_s^\infty \left[\frac{1}{s+1} - \frac{1}{s+3} \right] ds$$

$$= [\log(s+1) - \log(s+3)]_s^\infty = \left[\log \frac{(s+1)}{(s+3)} \right]_s^\infty$$

$$= 0 - \log \frac{(s+1)}{(s+3)} = \log \left[\frac{s+3}{s+1} \right]$$

$$\therefore \int_s^\infty e^{-st} \left[\frac{e^{-t} - e^{-3t}}{t} \right] dt = \log \left[\frac{s+3}{s+1} \right]$$

Taking as $s = 0$, then $\int_0^\infty e^{-st} \left[\frac{e^{-t} - e^{-3t}}{t} \right] dt = \log \left[\frac{0+3}{0+1} \right] = \log 3. \quad \text{Ans.}$

7. (a) Solve the D.E. $\frac{d^2 x}{dt^2} + x = t \cos 2t$

$$x(0) = x'(0) = 0$$

Solution. Here, $\frac{d^2x}{dt^2} + x = t \cos 2t$

Taking Laplace transform of both sides, we get

$$[s^2 \bar{x} - sx(0) - x'(0)] + \bar{x} = L[t \cos 2t]$$

$$[s^2 \bar{x} - sx(0) - x'(0)] + \bar{x} = \frac{-d}{ds} \left[\frac{s}{s^2 + 4} \right]$$

$$(s^2 + 1)\bar{x} = - \left[\frac{s^s + 4 - 2S^2}{(s^2 + 4)^2} \right] = - \left[\frac{4 - s^2}{(s^2 + 4)^2} \right]$$

or
$$\bar{x} = \frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)}$$

Taking inverse Laplace transform of both sides

$$x = L^{-1} \left[\frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)} \right] \quad \dots(i)$$

Let
$$\frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)} = \frac{z - 4}{(z + 4)^2 (z + 1)}, \text{ where } s^2 = z \quad \dots(ii)$$

Let
$$\frac{z - 4}{(z + 1)(z + 4)^2} = \frac{A}{(z + 1)} + \frac{B}{(z + 4)} + \frac{C}{(z + 4)^2} \quad \dots(iii)$$

$$z - 4 = A(z + 4)^2 + (z + 1)(z + 4) + C(z + 1) \quad \dots(iv)$$

Putting $z = -1$ in (4), $-5 = 9A \therefore A = \frac{-5}{9}$

Putting $z = -4$ in (4), $-8 = -3C \Rightarrow C = \frac{8}{3}$

Comparing the constant terms in (4)

$$-4 = 16A + 4B + C$$

$$-4 = \frac{-80}{9} + 4B + \frac{8}{3}$$

$$\frac{-36 - 24 + 80}{9} = 4B \Rightarrow B = \frac{5}{9}$$

$$\therefore \frac{s^2 - 4}{(S^2 + 4)^2 (s^2 + 1)} = \frac{-5}{9} \frac{1}{(s^2 + 1)} + \frac{5}{9} \frac{1}{(s^2 + 4)} + \frac{8}{3} \cdot \frac{1}{(s^2 + 4)^2}$$

$$\therefore L^{-1} \left[\frac{s^2 - 4}{(S^2 + 4)^2 (s^2 + 1)} \right] = \frac{-5}{9} \sin t + \frac{5}{9} \cdot \frac{1}{2} \sin 2t + \frac{8}{3} \cdot \frac{1}{2 \times 8} (\sin 2t - 2t \cos 2t)$$

$$= \frac{-5}{9} \sin t + \frac{5}{18} \sin 2t + \frac{1}{6} \sin 2t - \frac{1}{3} t \cos 2t$$

$$\left[\therefore \left[\frac{s}{s^2 + a^2} \right] = \frac{1}{2a} t \sin at \text{ and } L^{-1} \left[\frac{1}{(s^2 + a^2)^2} \right] = \frac{1}{2a^3} (\sin at - at \cos at) \right]$$

$$\left[\therefore L^{-1} \left[\frac{s}{s^2 + a^2} \right] \frac{1}{2 \times 8} = \frac{1}{2a^3} [\sin at - at \cos at] \right]$$

$$\therefore x = -\frac{5}{9} \sin t + \frac{4}{9} \sin 2t - \frac{1}{3} t \cos 2t$$

$$x = \frac{1}{9} [4 \sin 2t - 5 \sin t - 3t \cos 2t]. \text{ Ans.}$$

7. (b) Solve : $\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y.$

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Solution. Here, $\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y$

or $\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$

Intergrate w.r.t. x, keeping y as a constant.

$$\frac{\partial z}{\partial y} = e^y \cdot e^x + f(y)$$

Again Integrate w.r.t. y, we get

$$z = e^x e^y + \int f(y) dy + \phi(x)$$

$$z = e^x e^y + g(y) + \phi(x)$$

or $z = e^{x+y} + g(y) + \phi(x). \text{ Ans.}$

Where g(y) is the integral of f(y).

8. (a) Solve the D.E. $x^2 p + y^2 q = (x+y) z.$

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Solution. Here, $x^2 p + y^2 q = (x+y) z$

Comparing with $Pp + Qq = R$, we have

$$P = x^2, Q = y^2, R = (x+y)z$$

\therefore The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$

....(i)

Taking the first two members, we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

$$\frac{x^{-2+1}}{-2+1} - \frac{y^{-2+1}}{-2+1} = a$$

$$-\frac{1}{x} - \frac{1}{y} = a \quad \text{or} \quad \frac{1}{y} - \frac{1}{x} = a \quad \dots\dots(ii)$$

Again taking 1st and last members, we have

$$\frac{dx}{x^2} = \frac{dz}{(x+y)z}$$

$$\left(\frac{x+y}{x^2}\right)dx = \frac{dz}{z}$$

$$\left[\frac{1}{x} + \frac{y}{x^2}\right]dx = \frac{dz}{z}$$

$$\left[\frac{1}{x} + \frac{1}{x(ax+1)}\right]dx = \frac{dz}{z} \quad \left[\because y = \frac{x}{ax+1}\right] \quad \dots\dots(iii)$$

$$\text{Now} \quad \frac{1}{x(ax+1)} = \frac{A}{x} + \frac{B}{ax+1} \quad \text{or} \quad 1 = A(ax+1) + Bx$$

$$\text{put } x = -\frac{1}{a}, \quad 1 = \frac{-1}{a}B \Rightarrow B = -a$$

$$\text{put } x = 0, \quad 1 = A \quad \text{or } A = 1$$

$$\therefore \frac{1}{x(ax+1)} = \frac{1}{x} - \frac{a}{(ax+1)}$$

$$\text{From (iii), we get} \quad \left[\frac{1}{x} + \frac{1}{x} - \frac{a}{ax+1}\right]dx = \frac{dz}{z}$$

$$\text{Integrating on both sides} \quad 2 \int \frac{dx}{x} - a \int \frac{dx}{(ax+1)} = \int \frac{dz}{z}$$

$$2 \log x - a \times \frac{1}{a} \log(ax+1) = \log z + \log B$$

$$\log x^2 - \log \left[\frac{(x-y)}{xy} + 1 \right] - \log z = \log b \quad \left[\because a = \frac{x-y}{xy}\right]$$

$$\log x^2 - \log \frac{x}{y} - \log z = \log b$$

$$\log \left[\frac{x^2}{\frac{x}{y} \times z} \right] = \log b$$

$$\text{or} \quad b = \frac{xy}{z} \quad \dots(iv)$$

Combining (ii) and (i), the general solution is $\phi\left(\frac{xy}{z}, \frac{1}{y} - \frac{1}{x}\right) = 0$. **Ans.**

8. (b) Solve using Charpit's method $(p^2 + q^2)x = pz$

Solution Here, $f(x, y, z, p, q) = (p^2 + q^2)x - pz = 0$

.....(i)

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$$\therefore \frac{\partial f}{\partial x} = p^2 + q^2, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = -p, \frac{\partial f}{\partial p} = 2px - z, \frac{\partial f}{\partial q} = 2qx$$

Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\text{or } \frac{dx}{-(2px - z)} = \frac{dy}{-2qx} = \frac{dz}{-p(2px - z) - q(2qx)} = \frac{dp}{p^2 + q^2 - p^2} = \frac{dq}{0 + q \times -p}$$

$$\frac{dx}{-2px + z} = \frac{dy}{-2qx} = \frac{dz}{-2p^2x + pz - 2q^2x} = \frac{dp}{q^2} = \frac{dq}{-pq}$$

From last two member of (i), we have

$$\frac{dp}{q^2} = \frac{dq}{-pq} \quad \text{or} \quad \int \frac{dp}{q} = \int \frac{dq}{-p} \Rightarrow$$

$$\int p dp = - \int q dq \Rightarrow \frac{p^2}{2} = \frac{-q^2}{2} + \frac{a^2}{-p}$$

Integrating $p^2 = -q^2 + a^2$ (say)

$$\text{or } q^2 + p^2 = a^2$$

Using (iii) in (i), we have

$$a^2x - pz = 0$$

$$p = \frac{a^2x}{z}$$

Now

$$q^2 + p^2 = a^2$$

or

$$q^2 = a^2 - \frac{a^4x^2}{z^2}$$

or

$$q = \frac{a}{z} \sqrt{z^2 - a^2x^2}$$

Now $dz = p dx + q dy$

Substituting the values of p and q, we have

$$dz = \frac{a^2x}{z} dx + \frac{a}{z} \sqrt{z^2 - a^2x^2} dy$$

$$zdz = a^2x dx + a\sqrt{z^2 - a^2x^2} dy$$

or

$$zdz - a^2x dx = a\sqrt{z^2 - a^2x^2} dy$$

$$\frac{zdz - a^2x dx}{\sqrt{z^2 - a^2x^2}} = a dy$$

$$\frac{\frac{1}{2}d(z^2 - a^2x^2)}{\sqrt{z^2 - a^2x^2}} = a dy$$

Integrating
$$\frac{\frac{1}{2}(z^2 - a^2x^2)^{-\frac{1}{2}+1}}{\frac{1}{2}} = a y + b$$

$$\sqrt{z^2 - a^2x^2} = (ay + b)$$

$$\therefore z^2 = (ay + b)^2 + a^2x^2$$

Which is the required complete integral. **Ans.**

8. (c) Solve $x(1+y)p = y(1+x)q$

Solution. Given eq. is $x(1+y)p = y(1+x)q$

or
$$\frac{xp}{(1+x)} = \frac{yq}{(1+y)}$$

which is of the form $f_1(x, p) = f_2(y, q)$

Let
$$\frac{xp}{1+x} = \frac{yq}{(1+y)} = a$$

then
$$\frac{xp}{(1+x)} = a \quad \text{or} \quad p = \frac{(1+x)a}{x}$$

and
$$q = \frac{(1+y)a}{y}$$

These value putting in $dz = x dx + y dy$

$$dz = \frac{a(1+x)}{x} dx + a \frac{(1+y)}{y} dy$$

Integrating

$$\int dz = a \int \left(\frac{1}{x} + 1 \right) dx + a \int \left(\frac{1}{y} + 1 \right) dy$$

$$z = a(\log x + x) + a(\log y + y) + b$$

$$z = a[\log x + \log y] + a(x + y) + b$$

$$\log(xy) + a(x + y) + bz = a$$

which is the required complete solution.